The Single-Cycle Scheme: A New Approach to Numerical Optimization

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A new scheme is presented for solving optimization problems, in which the objective function is dependent on the solution of a partial differential equation. The scheme can be obtained by modifying any standard iterative procedure for solving the partial-differential equation. This modified procedure, which updates the solution of the differential equation and the design parameters simultaneously, eliminates the need for the costly inner-outer iterative procedure. The scheme is demonstrated by application to the problem of determining wind tunnel wall interference corrections. Results indicate that the ratio of the cost of solving the optimization problem to the cost of solving the partial-differential equation using a standard iterative scheme is less than $L+1$, where L is the number of design parameters.

Nomenclature

c_1 = incrementing factor for optimization scheme
c_2 = decrementing factor for optimization scheme
E = objective function (or the performance index)
L = number of design parameters
M = Mach number
N_e = number of iterative sweeps required for updating ϕ by the optimization scheme
N_P = number of iterative sweeps required for updating P by the optimization scheme
N_s = number of iterative sweeps required to solve Eq. (2), with $P=P^*$, using a standard iterative procedure.
N_t = number of iterative steps required for solving Eqs. (1) and (2) using the optimization scheme
N_w = number of iterative sweeps required for solving Eqs. (1) and (2) using the optimization scheme
$\boldsymbol{P}$ = vector of design parameters (or the decision vector)
R = residual
x = coordinate in the undisturbed flow direction
X = x coordinate relative to the wing leading edge
y = coordinate normal to the model plane of symmetry
Y = y coordinate normalized by the wing semispan
z = vertical coordinate in the upward direction
α = angle of attack
γ = ratio of specific heats
ΔN = number of iterative steps after which the value of $\boldsymbol{P}$ is periodically updated
ΔP = small positive incremental value
δP = incremental value used to update the design parameter
ϵ = small positive number determining convergence
κ_e = N_e/N_s
κ_p = N_p/N_s
μ = $N_w/(L+1)N_s$
ρ = density
τ = maximum wing thickness
ϕ = perturbation velocity potential

Subscripts

F = free air
FS = model surface in free air
i = initial iterative value
l = element number in design parameter vector
m = outer iteration number
T = tunnel
TS = model surface in tunnel
∞ = undisturbed condition

Superscripts

c = coarse computational mesh
f = fine computational mesh
n = inner iteration number
$*$ = optimum value

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Introduction

THIS paper presents a new optimization scheme for solving the following optimization problem. Find the optimum value of $\boldsymbol{P}$, $\boldsymbol{P}^*$, for which

$$E(\boldsymbol{P}^*;\phi) = \min_{\boldsymbol{P}} E(\boldsymbol{P};\phi) \tag{1}$$

with ϕ satisfying the partial-differential equation

$$D(\phi;\boldsymbol{P}) = 0 \tag{2a}$$

subject to the boundary condition

$$B(\phi;\boldsymbol{P}) = 0 \tag{2b}$$

The approach used here applies to problems for which the solution $\phi(x,y,z)$ to the boundary-value problem, Eqs. (2), for a given value of $\boldsymbol{P}$, is obtained iteratively. An iterative solution to Eqs. (2) may be required if Eq. (2a) is nonlinear, if the boundary condition is nonlinear, if Eq. (2a) is linear with variable coefficients, and so on. The same approach can be applied to problems where the single equation (2a) is replaced by a system of equations; however, the discussion here is confined to the case of a single partial-differential equation.

The optimization scheme presented here is demonstrated by application to examples from the field of aerodynamics. The area of wind tunnel wall interference corrections is an example of where the optimization problem, Eqs. (1) and (2), arises.[1] There, $\boldsymbol{P}$ may be chosen to include the free-air Mach number and angle of attack. The problem then is to find the optimum value of $\boldsymbol{P}$ that minimizes E, where E is chosen to be a measure of the difference between the Mach number on the model surface in the tunnel and that in free air. In the aerodynamic applications discussed here, Eq. (2a) is the potential flow equation or an approximation of it. Airfoil and wing design problems are other areas in which the optimization problem, Eqs. (1) and (2), arises.[2-4] In these cases,

P contains the coefficients of the polynomials or the shape functions used to define the lifting surface, and E is the drag, the negative lift, or a measure of the difference between the pressure on the lifting surface and a desired pressure distribution.

The procedures presently available for solving the optimization problem[2-4] are time consuming, and their high cost prevents their regular use. These procedures are inner-outer iterative procedures. In such procedures, each outer iteration results in a new iterative solution for P by using an optimization scheme (e.g., the steepest descent method or the conjugate gradient method). However, each outer iteration requires that the objective function be estimated and, therefore, that a solution for Eq. (2a) be obtained (e.g., by line relaxation) using the new P. Consequently, Eq. (2a) must be solved many times before the optimum P value is determined, resulting in the high cost of calculation. A reduction in the number of times Eq. (2a) must be solved before the optimum P value is determined has been achieved recently by utilizing fast approximation methods.[5,6]

The optimization scheme presented here eliminates the need for an inner-outer iterative procedure and, thus, the boundary value problem, Eqs. (2), only needs to be solved once. The approach used was first presented in Ref. 7. The optimization scheme developed there, however, is only applicable to single-parameter problems (i.e., P is a scalar) with an objective function of a special form. Here, a general scheme applicable to multi-design-parameter problems is developed and tested.

Formulation

The Old Approach

Available optimization schemes determine the optimum parameter P^* that satisfies condition (1) through an iterative process. A number of outer iterations or iterative cycles are required to complete the iterative process. The iterative cycles determine a sequence of successive approximations, P_m, where $m=1,2,\ldots$, that converge to the optimum value P^*. Within the mth iterative cycle (outer iteration), the boundary value problem, Eqs. (2), is solved by making a number of iterative sweeps (inner iterations). The inner iterative sweeps update the value of ϕ, giving the successively improved approximations, ϕ_m^n, where $n=1,2,\ldots$, which converge to the solution ϕ_m, satisfying the boundary value problem

$$D(\phi_m;P_m)=0 \qquad m=1,2,\ldots$$

$$B(\phi_m;P_m)=0 \qquad m=1,2,\ldots$$

Figure 1 shows the value of E as a function of the value of P over a number of iterations. The lines shown on the surface are constant n lines and constant P lines. Each of the constant P lines shows the history of E as a function of n for a given P value. Curve AB traces the minimum values of E as a function of n. In the old approach to optimization, a value of P is chosen and the analysis boundary value problem, Eqs. (2), is solved, tracing a constant P curve on surface E in Fig. 1 as the iterative solution is obtained. If the value of E when convergence is reached does not lie on curve AB, a new P value is chosen and another analysis boundary value problem is solved, tracing another constant P curve on surface E. This procedure continues until a P value is found for which E lies on curve AB when convergence is reached.

The New Approach

Unlike the old approach just described for solving Eqs. (1) and (2), the approach followed here eliminates the need for the outer iterative cycles. In this approach, the boundary-value problem (2) is solved in an iterative manner that updates the values of P as well as ϕ in each iterative sweep. This results in the successively improved approximations (ϕ^n, P^n), $n=1,2,\ldots$, which converge to the solution (ϕ^*, P^*), satisfying the optimization problem, Eqs. (1) and (2). This basic idea of updating both ϕ and P simultaneously can be used to develop a family of efficient optimization schemes. A single-parameter scheme belonging to this family was developed in Ref. 7. Below, a general scheme applicable to multi-parameter problems is developed. A discussion of the newly developed scheme is given for the special case of a single-design-parameter problem. The equations are developed for this special case, then they are generalized to multi-design-parameter problems.

Modifying the iterative procedure for the analysis problem to allow the value of P to vary as the iterations proceed results in a family of E curves similar to those of Fig. 1. One of these curves defines the minimum E curve CD (see Fig. 1) similar to the minimum curve AB. The projection, $C'D'$, of the CD curve onto the P-n plane defines the minimum E path given by

$$P=P^n=P^n_{\min}$$

The new scheme developed here modifies the iterative procedure for solving the analysis problem to follow a modified minimum E path

$$P^n=P^n_{a\,\min}$$

which is an approximation to the minimum E path. Since this path is not known a priori, it must be determined during the iterative procedure. Iterative sweeps are required to determine the path $P^n_{a\,\min}$. It is therefore noted that, in the analysis iterative procedure, given the nth iterative solution ϕ^n, the $n+1$st iterative step, which determines the iterative solution ϕ^{n+1}, requires only a single iterative sweep defined by

$$\phi^{n+1}=\psi(\phi^n,P)$$

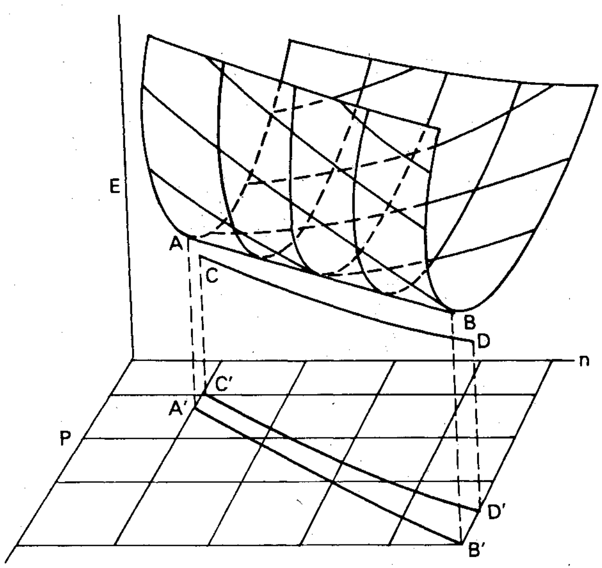

Fig. 1 Dependence of objective function history on P.

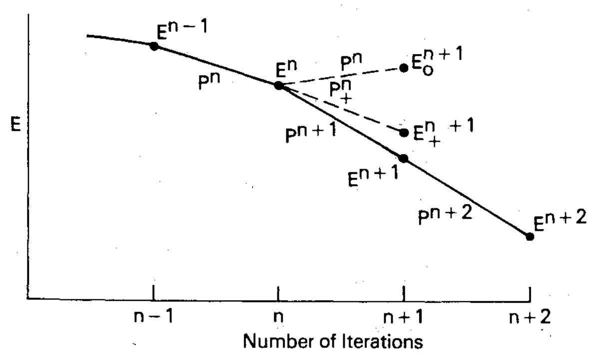

Fig. 2 Optimization scheme.

where $\psi(\phi^n,P)$ denotes the solution obtained by making a single iterative sweep using ϕ^n as an initial guess and P as the value of the design parameter. Similarly, in the new scheme, given the nth iterative solutions (ϕ^n,P^n), the $n+1$st iterative step, which determines the iterative solutions ϕ^{n+1} and P^{n+1}, requires a single iterative sweep to evaluate ϕ^{n+1},

$$\phi^{n+1}=\psi(\phi^n,P^{n+1})$$

However, to determine P^{n+1}, further iterative sweeps may be required. Following the minimum E path ($P^n=P^n_{\min}$) would be costly since the precise determination of such a path would require many iterative sweeps each iterative step. It is for this reason that the approximate minimum path, $P^n_{a\,\min}$, has been chosen. This path is chosen such that the deviation between the paths $P^n_{a\,\min}$ and $P^n_{\min}$ is proportional to the deviation between the iterative solution ϕ^n and the final converged solution ϕ^*. In other words, for small values of n, for which ϕ^n is not a good approximation to ϕ^*, the deviation between the two paths is allowed to be large; however, for large n values (for which ϕ^n is nearly converged) both paths converge.

In the $n+1$st iterative step, the value of P is updated by the equation

$$P^{n+1}=P^n+\delta P^{n+1}$$

where

$$\delta P^{n+1}=\tfrac{1}{2}\delta P^\eta\,[c_1(s^{n+1}+1)+c_2(s^{n+1}-1)],\qquad \frac{n+1}{\Delta N}=1,2,3,\ldots$$

$$=0,\qquad \frac{n+1}{\Delta N}\neq 1,2,3,\ldots$$

$$s^{n+1}=\frac{\delta P^\eta\,(E_0^{n+1}-E_+^{n+1})}{|\delta P^\eta\,(E_0^{n+1}-E_+^{n+1})|}$$

$$E_0^{n+1}=E(P^n;\Phi^{0,n+1}),\qquad E_+^{n+1}=E(P_+^n;\Phi^{1,n+1})$$

$$\Phi^{0,n+1}=\psi(\phi^n,P^n),\qquad \Phi^{1,n+1}=\psi(\phi^n,P_+^n)$$

$$P_+^n=P^n+\Delta P,\qquad \eta=n+1-\Delta N$$

and ΔP is a small, positive perturbation. Since updating P every time ϕ is updated may be too costly, the scheme allows P to be updated every ΔN iterative steps, where $\Delta N\geq 1$. The sign of the increment δP^{n+1} is taken to be the same as the sign of $(E_0^{n+1}-E_+^{n+1})$, while the magnitude of δP^{n+1} is $c_1\,|\delta P^\eta|$ and $c_2\,|\delta P^\eta|$ for the cases in which δP^{n+1} and δP^η are of the same sign and opposite signs, respectively. The constants c_1 and c_2 are positive with $c_1>1$ and $c_2<1$. The $n+1$st iterative solution for ϕ is given by

$$\phi^{n+1}=\psi(\phi^n,P^{n+1}) \tag{3}$$

The following formula allows the calculation of ϕ^{n+1} without performing the iterative sweep defined by Eq. (3):

$$\phi^{n+1}=\Phi^{0,n+1}+\frac{\Phi^{1,n+1}-\Phi^{0,n+1}}{\Delta P}\,\delta P^{n+1}$$

The iterative process continues until the convergence criterion

$$\max(\,|R^n|-\epsilon_e,\ |\delta P^n|-\epsilon_P)<0 \tag{4}$$

is satisfied, where R^n is the maximum residual for the set of finite-difference equations at the nth iteration, and ϵ_e and ϵ_P are small positive numbers. The iterative process is depicted in Fig. 2.

In extending the scheme to a multiparameter problem, it becomes necessary to update each of the design parameters. To update a given design parameter, the values from the previous iterative step of the other design parameters are held fixed and the scheme described for the case of a single-parameter problem is used. In the $n+1$st iterative step, the value of P_l (the lth element of the design parameter vector) is updated by

$$P_l^{n+1}=P_l^n+\delta P_l^{n+1}\qquad l=1,2,\ldots,L$$

where

$$\delta P_l^{n+1}=\tfrac{1}{2}\delta P_l^\eta\,[c_1(s_l^{n+1}+1)+c_2(s_l^{n+1}-1)],\qquad \frac{n+1}{\Delta N}=1,2,3,\ldots$$

$$=0,\qquad \frac{n+1}{\Delta N}\neq 1,2,3,\ldots$$

$$s_l^{n+1}=\frac{\delta P_l^\eta\,[E(\boldsymbol{P}^{0,n};\Phi^{0,n+1})-E(\boldsymbol{P}^{l,n};\Phi^{l,n+1})]}{|\delta P_l^\eta\,[E(\boldsymbol{P}^{0,n};\Phi^{0,n+1})-E(\boldsymbol{P}^{l,n};\Phi^{l,n+1})]|}$$

$$\Phi^{0,n+1}=\psi(\phi^n,\boldsymbol{P}^{0,n}),\qquad \Phi^{l,n+1}=\psi(\phi^n,\boldsymbol{P}^{l,n})$$

$$\boldsymbol{P}^{0,n}=\begin{bmatrix}P_1^n\\ P_2^n\\ \vdots\\ P_L^n\end{bmatrix}$$

$$\boldsymbol{P}^{l,n}=\boldsymbol{P}^{0,n}+\boldsymbol{e}^l\Delta P\qquad l=1,2,\ldots,L$$

$$\boldsymbol{e}^l=\begin{bmatrix}e_1\\ e_2\\ \vdots\\ e_L\end{bmatrix}$$

and

$$e_j=0\quad j\neq l,\qquad e_j=1\quad j=l$$

The $n+1$st iterative solution for ϕ is then given by

$$\phi^{n+1}=\psi(\phi^n,\boldsymbol{P}^{0,n+1}) \tag{5}$$

The following formula allows the calculation of ϕ^{n+1} without performing the iterative sweep defined by Eq. (5):

$$\phi^{n+1}=\Phi^{0,n+1}+\sum_{l=1}^{L}\frac{\Phi^{l,n+1}-\Phi^{0,n+1}}{\Delta P}\,\delta P_l^{n+1} \tag{6}$$

The iterative process continues until the convergence criterion

$$\max(\,|R^n|-\epsilon_e,\ |\delta P_l^n|-\epsilon_P)<0\qquad l=1,2,\ldots,L$$

is satisfied.

Results and Discussion

In this section, the optimization procedure described previously is applied to examples that arise from wind tunnel wall interference correction problems. The correction procedure of Ref. 1 is formulated as an optimization problem with constraints. In order to test the new optimization scheme, a modification of the procedure of Ref. 1, which reformulates the problem as an optimization problem without constraints, is used. An inviscid flow about the tested model in the tunnel is numerically simulated and, through an iterative procedure described in Ref. 1, the angle of attack α_T is determined such that the calculated lift is equal to the measured lift. The angle of attack will differ from the actual angle of attack of the tested model due to viscous effects present in the experiment, but not in the numerical simulation of the flow and to the geometrical differences between the actual model and the numerical simulation of the model. The Mach number distribution on the wind tunnel model surface, M_{TS}, at an angle of attack α_T and for a freestream Mach number $M_{T\infty}$ equal to the experimental freestream Mach number is calculated. The corrected free-air angle of attack α_F^* and Mach number $M_{F\infty}^*$ are found for the inviscid flow by solving Eq. (1) with

$$E(P;\phi)=\int [M_{FS}(P;\phi)-M_{TS}]^2 \mathrm{d}S \Big/ \int M_{TS}^2 \mathrm{d}S$$

The integrals here are taken over the model surface. The objective function is a measure of the Mach number difference on the model surface for the numerically simulated free-air flow and the numerically simulated wind tunnel flow. The flow is assumed to be governed by the transonic small-disturbance equation. Therefore,

$$D(\phi;P)\equiv[\rho(1+\phi_x)]_x+\phi_{yy}+\phi_{zz}$$

where

$$\rho=1-\phi_x+(1-M_\infty^2)\phi_x+[(1-\gamma)/2]\phi_x^2+\ldots$$

Equation (2a) is solved subject to the model surface boundary condition specifying zero flow through the model surface and to free-air far-field conditions. The successive line overrelaxation procedure[8] for solving the transonic small-disturbance equation is modified to allow the updating of the design parameters as described above.

The wind tunnel correction procedure is applied to an airfoil in a solid-wall tunnel (see Fig. 3a) and to a wing in a solid-wall tunnel (see Figs. 3a and 3b). The optimization procedure is applied first to the problem of correcting the freestream Mach number due to wind tunnel blockage effects. In this problem, the design parameter P is a scalar equal to the undisturbed free-air Mach number. The optimum value of P is equal to the corrected free-air Mach number $M_{F\infty}^*$. Mach number corrections are calculated for a NACA 0012 airfoil of chord 1.0 and zero angle of attack located in the middle of a solid-wall wind tunnel of height 3.26 (see Fig. 3a). The wind tunnel freestream Mach number $M_{T\infty}$ is taken to be 0.8.

A comparison between the Mach number distribution on the airfoil surface for the wind tunnel flow ($M_\infty=0.8$), the free-air flow at the corrected condition ($M_\infty=0.8174$), and the free-air flow at the uncorrected condition ($M_\infty=0.8$) is given in Fig. 4. The objective function was reduced from the value $E=3.46\times10^{-3}$ for the flow with the uncorrected Mach number ($M_\infty=0.8$) to the value $E=1.39\times10^{-5}$ for the flow with the corrected Mach number ($M_\infty=0.8174$).

To estimate the effect of correcting the Mach number, as described above, on the rate of convergence of the flowfield solution, a standard calculation with $M_{F\infty}=0.8174$ is performed. In this calculation, $M_{F\infty}$ is not updated during the iterative process. A solution obtained on a coarse computational mesh is used as an initial guess for solving the problem on the final mesh. A comparison between the number of iteration sweeps (N_w^c and N_w^f) required for the convergence ($\epsilon_e^c=0.5\times10^{-3}$, $\epsilon_e^f=0.5\times10^{-3}$, $\epsilon_P^c=0.5\times10^{-4}$, $\epsilon_P^f=0.5\times10^{-5}$) of the present iterative process and the number of iterative sweeps ($N_s^c=133$, $N_s^f=181$) required for the convergence ($\epsilon_e^c=0.5\times10^{-3}$, $\epsilon_e^f=0.5\times10^{-3}$) of the standard iterative process is made for different parametric values. Before presenting the results of the parametric study, it is convenient to define μ,

$$\mu=\frac{N_w}{N_s(L+1)}\doteq\frac{N_t(1+L/\Delta N)}{N_s(L+1)}$$

to be a measure of the scheme efficiency. The efficiency is increased as the value of μ is decreased. The total number of iterative sweeps N_w may be expressed as the sum

$$N_w=N_e+N_P$$

of the number of iterative sweeps $N_e(=N_t)$ required to solve the partial-differential equation and the number of iterative sweeps $N_p(\doteq N_tL/\Delta N)$ required to determine the approximate minimum E path

$$P^n=P^n{}_{a\,\mathrm{min}}$$

Unless otherwise stated, the following parametric values are used:

$$c_1=1.2,\qquad c_2=0.6$$
$$\delta P_i^c=0.02,\qquad \delta P_i^f=0.002$$
$$P_i^c=0.83,\qquad \Delta N=1$$

and P_i^f is always taken to be equal to the optimum P value as determined from the coarse mesh calculation. The value of ΔP is taken to be 10^{-6} in all calculations.

Three sets of values for c_1 and c_2 were used in the calculations. Table 1 indicates that the scheme efficiency is insensitive to variations in c_1 and c_2. Since $\mu^c<1$ and $\mu^f<1$, the minimum E path increases the rate of convergence for solving Eqs. (2) compared to the corresponding rate of the standard iterative path procedure in which a constant P path ($P=P^*$) is followed. Similar conclusions can be drawn from Tables 2 and 3, in which the values of δP_i^c and P_i^c have been varied. Table 3 also indicates that the scheme has good convergence properties for the case in which the initial guess for the design parameter is chosen to closely approximate the optimum solution, as well as the case in which this initial guess is chosen away from the optimum solution. Therefore, the present scheme does not have the difficulties[9] associated

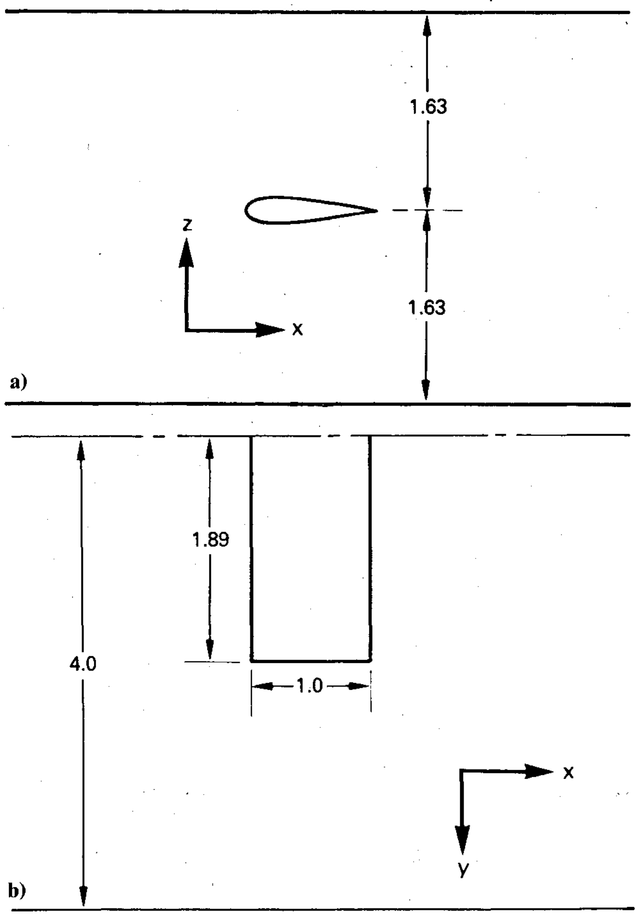

Fig. 3 Wing in tunnel: a) side view; b) top view.

Table 1 Effect of c_l and c_2 on scheme efficiency

c_l	c_2	μ^c	μ^f
1.1	0.8	0.83	0.90
1.2	0.6	0.82	0.90
1.5	0.4	0.83	0.90

Table 2 Effect of δP_i on scheme efficiency

δP_i^c	δP_i^f	μ^c	μ^f
0.002	0.002	0.82	0.90
0.02	0.002	0.82	0.90
0.2	0.002	0.82	0.90

Table 3 Effect of P_i on scheme efficiency

P_i^c	P_i^f	μ^c	μ^f
0.4	0.8154	0.83	0.90
0.83	0.8154	0.82	0.90
0.90	0.8154	0.82	0.90

Table 4 Effect of ΔN on scheme efficiency

ΔN	κ_e^c	κ_p^c	κ_e^f	κ_p^f	μ^c	μ^f
1	0.82	0.82	0.90	0.90	0.82	0.90
2	0.82	0.41	0.92	0.46	0.61	0.69
3	0.81	0.27	0.96	0.32	0.54	0.64
4	0.83	0.20	0.94	0.23	0.52	0.59
5	1.17	0.23	0.97	0.19	0.70	0.58
6	1.08	0.18	1.13	0.19	0.63	0.66
7	1.21	0.17	1.62	0.23	0.69	0.93

with first-order gradient methods (which have poor convergence characteristics as the optimum solution is approached) and with second-order gradient methods (which may have convergence problems away from the optimum solution).

Table 4 shows the effect of varying ΔN on the scheme efficiency. In addition to the values of μ, the values of κ_e and κ_p are given. While μ is a measure of the scheme efficiency, κ_e is a measure of the cost of determining the solution ϕ as the approximate minimum path ($P^n = P^n_{a\,\min}$) is followed, and κ_p is a measure of the cost required to determine this path. It is noted that there is an optimum ΔN value, ΔN^*, which causes μ to take its minimum value. The table indicates that the optimum value is 4 for the coarse mesh calculations and 5 for the fine mesh calculations. This optimum value for ΔN is that value which causes the convergence criteria

$$|R^n| - \epsilon_e < 0 \tag{7}$$

and

$$|\delta P^n| - \epsilon_P < 0 \tag{8}$$

to be satisfied simultaneously. In this case, both ϕ^* and P^* will be determined to the desired accuracy. However, if $\Delta N < \Delta N^*$, condition (8) becomes satisfied at an earlier iteration than that at which condition (7) becomes satisfied. To satisfy the convergence criterion, Eq. (4), the scheme will determine P^* to a higher order of accuracy than desired, leading to unnecessary computational costs. If $\Delta N > \Delta N^*$, the reverse situation occurs, and the scheme determines ϕ^* to a higher order of accuracy than required.

Table 5 shows the results of varying c_l and c_2 for a ΔN value of 4. A comparison of Tables 1 and 4 indicates that the scheme efficiency is less sensitive to variations in c_l and c_2 for the case with $\Delta N=1$ than for the case with $\Delta N=4$. This is

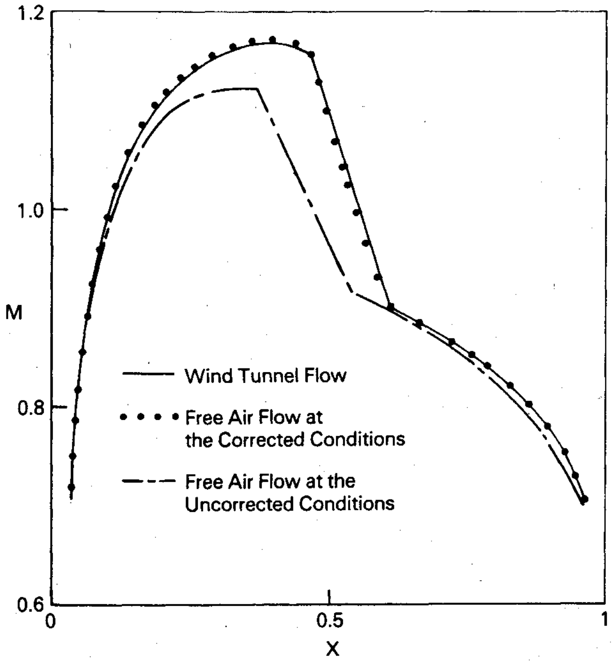

Fig. 4 Mach number distribution on airfoil surface.

expected, since the path $P = P_{a\,\min}$ is updated more often in the first case than in the second.

A multi-design-parameter example has been calculated with

$$P = \begin{bmatrix} M_{F\infty} \\ \tau \end{bmatrix}, \qquad P_i = \begin{bmatrix} 0.83 \\ 0.13 \end{bmatrix}$$

The optimum solution for this example was found to be given by

$$P^* = \begin{bmatrix} 0.8181 \\ 0.1193 \end{bmatrix}$$

Allowing τ to be a design parameter has slightly improved the value of $E = 1.29 \times 10^{-5}$ over the one design-parameter case ($E = 1.39 \times 10^{-5}$). Table 6 compares the scheme efficiency for the two cases.

The correction procedure has been applied to a rectangular wing with a NACA 0012 airfoil section. The geometrical configuration of the wing in a solid-wall wind tunnel is shown in Figs. 3a and 3b. The undisturbed tunnel Mach number is assumed to be 0.8, while the value of the lift experienced by the wing is assumed to be 0.583. The corresponding angle of attack α_T is found to be 2.6765 deg. In this problem, the vector P is defined by

$$P = \begin{bmatrix} M_{F\infty} \\ \tau \\ \alpha_F \end{bmatrix}$$

The optimization problem is solved with

$$P_i = \begin{bmatrix} 0.4 \\ 0.2 \\ 2.0 \end{bmatrix}, \qquad \delta P_i^c = \begin{bmatrix} 0.1 \\ 0.02 \\ 0.5 \end{bmatrix}$$

$$\delta P_i^f = \begin{bmatrix} 0.005 \\ 0.005 \\ 0.10 \end{bmatrix}, \qquad \Delta N = 3$$

Comparisons between the Mach number distributions on the wing top and bottom surfaces at the mid-semispan station for the wind tunnel flow ($M_\infty = 0.8$, $\tau = 0.12$, $\alpha = 2.7675$ deg), the free-air flow at the corrected conditions ($M_\infty = 0.8130$, $\tau = 0.1169$, $\alpha = 3.2136$ deg), and the free-air flow at the uncorrected conditions ($M_\infty = 0.8$, $\tau = 0.12$, $\alpha = 2.7675$ deg) are given in Figs. 5 and 6. Figure 7 compares the shock wave location $[X = X_s(y)]$ for the three flows.

The objective function was reduced from the value $E = 2.73 \times 10^{-3}$ for the flow at the uncorrected conditions to the value $E = 1.21 \times 10^{-4}$ for the flow with the corrected conditions. Table 7 compares the efficiency of this example to the two-dimensional one ($P = M_{F\infty}$) and two $[P = (M_{F\infty}, \tau)]$ design-parameter problems. All three examples have been calculated using the same scheme parameter values.

The present optimization scheme has been applied to the problem of correcting wind tunnel wall interference effects, while the old approach has been applied to airfoil design problems[2] and wing design problems.[3,4] Also, the more recent approximation methods[5,6] have been applied to airfoil design problems. It is, therefore, not possible to make a comparison between the results of the present approach and those of previous approaches. Nevertheless, it is appropriate to give a limited discussion about certain aspects of the different approaches.

In applying the old approach to airfoil and wing design problems, the usual procedure has been to couple an existing aerodynamic analysis code (which solves Eqs. (2) iteratively for a given P) to an optimization code (which finds the optimum P iteratively). The repetitive execution of time-consuming analysis codes is the source of the high cost of the old approach. In Refs. 5 and 6, the number of the times the analysis code must be executed is reduced by evaluating the objective function using approximation methods.

The approach used in Ref. 5 is to develop a second-order Taylor series expansion of the function in terms of the design parameters. The coefficients of the expansion are calculated using a least-square fit to data developed earlier in the optimization process. The Taylor series expansion is then used during optimization, rather than calling the aerodynamics program for precise determination of the objective function. When the optimum airfoil has been found, based on this approximation, the airfoil is analyzed precisely by using the aerodynamics program. The results are added to the data set, and a new approximation to the objective function is defined. Optimization is then performed using the new function, and the process is repeated until convergence is achieved. To provide a second-order Taylor series expansion, $(L+1)(1+L/2)$, separate analyses are required. Therefore, assuming that only a few analyses are required beyond those needed for the second-order approximation, this method requires fewer aerodynamic analyses than the old method, provided the number of design parameters is less than 20. In a typical design, the old approach requires about 10 $(L+3)$ aerodynamic analyses to achieve an optimum design.[5] In both the old approach and the approach of Ref. 5, the number of aerodynamic analyses required to achieve an optimum design may deviate widely from the above estimates, depending on the initial estimate of the solution at the beginning of the

Table 5 Effect of c_1 and c_2 on scheme efficiency for $\Delta N = 4$

c_1	c_2	μ^c	μ^f
1.1	0.8	1.02	0.81
1.2	0.6	0.52	0.59
1.5	0.4	0.51	0.68

Table 6 Effect of L on scheme efficiency

L	κ_e^c	κ_p^c	κ_e^f	κ_p^f	μ^c	μ^f
1	0.82	0.82	0.90	0.90	0.82	0.90
2	0.68	1.37	0.86	1.72	0.68	0.86

Table 7 Scheme efficiency for two- and three-dimensional examples

Example	L	μ^c	μ^f
2-D	1	0.58	0.62
2-D	2	0.90	0.49
3-D	3	0.82	0.70

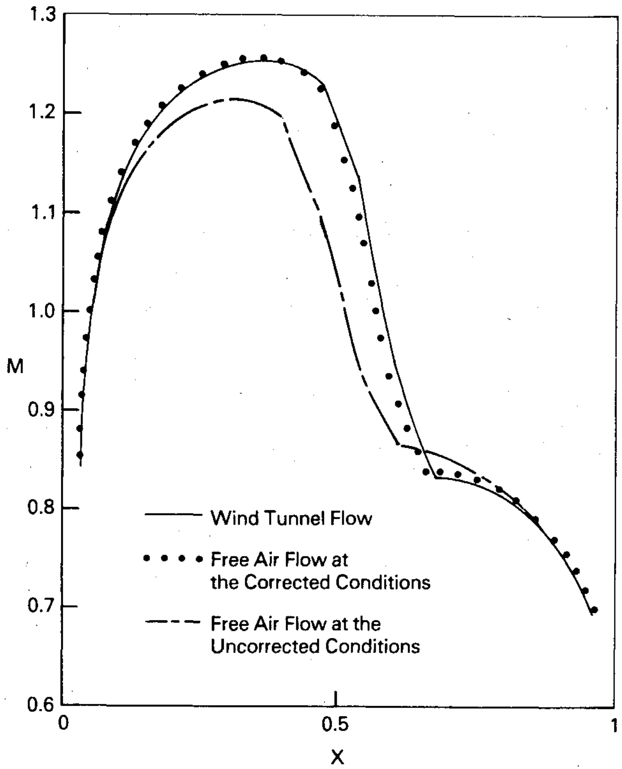

Fig. 5 Mach number distribution on wing top at the mid-semispan station.

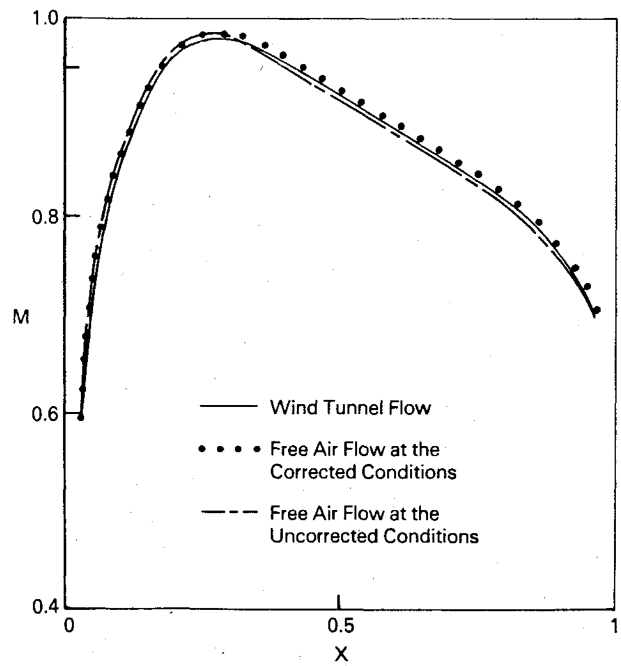

Fig. 6 Mach number distribution on wing bottom at the mid-semispan station.

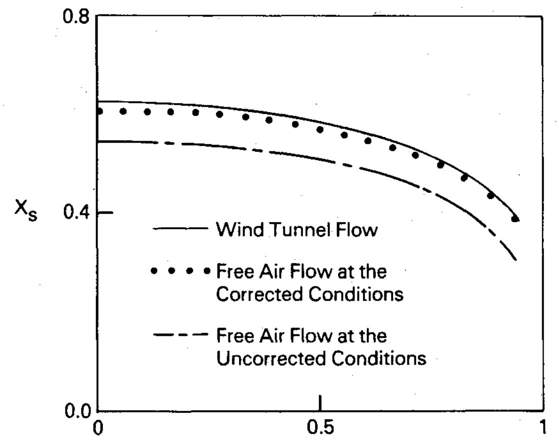

Fig. 7 Shock position on wing top.

design process. While an initial estimate of the solution, which closely approximates the optimum solution, will require approximately $(L+1)(1+L/2)$ separate analyses to achieve an optimum solution by the method of Ref. 5, a much larger number of separate analyses will be required for an initial estimate of the solution chosen independently from the optimum solution.

The idea of using approximate functions during optimization, instead of calling the aerodynamics program for precise determination of the objective function, has been used in developing the optimization scheme of Ref. 6. This scheme uses the method of coordinate straining to evaluate the approximate solutions. In this method, the approximate determination of perturbations in solutions that correspond to perturbations in some parameters are evaluated using two baseline solutions. The baseline solutions are evaluated using the aerodynamics program. The optimization procedure begins by making $(L+1)$ aerodynamic analyses to allow approximate evaluations of objective function perturbations corresponding to design parameter perturbations. Approximate solutions are then used during optimization. When the optimum airfoil has been found, based on this approximation, the airfoil is analyzed precisely by using the aerodynamics program. The program is then called to make L analyses to allow the evaluation of new approximate solutions, and the process is repeated until convergence is achieved. The total number of analyses is, therefore, $K(L+1)$, where K is the number of cycles required for convergence. This number may vary widely depending on the initial estimate of the solution at the beginning of the design process.

The use of the method of coordinate straining in the optimization procedure of Ref. 6 limits its applicability to a special class of nonlinear problems. Straining the coordinates allows the approximate solutions to include the nonlinear behavior resulting from the movement of discontinuities or high-gradient regions. However, if the locations of these special regions are unaffected by design parameter perturbations, the resulting approximate solutions coincide with linear interpolation, and the objective function approximation becomes a linear function of the design parameters. The solution to the approximate optimization problem is then unbounded, leading to no meaningful results. Moreover, if the baseline solutions contain no discontinuities or special high-gradient regions, the coordinate straining method becomes inapplicable.

For the examples solved here using the new scheme, the number of equivalent analyses required to achieve an optimum design is equal to $\mu(L+1)$ where $0.49 \le \mu \le 1.02$. An equivalent analysis is defined to be that analysis process which solves the boundary-value problem (2) with $\boldsymbol{P}$ set equal to $\boldsymbol{P}^*$ and with an initial estimate of the solution equal to that used at the beginning of the optimization process. Although some estimates have been presented for the number of times the analysis program is executed using different optimization schemes, a comparison between the relative costs of the different optimization procedures is not possible at this time. In comparing previous procedures, authors have used the number of times the analysis program is executed during the design process as a measure of the cost. This, however, is not a good measure for comparison since the cost per execution may vary widely, depending on the initial estimate of the solution at the beginning of the execution. A better measure for comparison between the different procedures is N_w, the total number of iterative sweeps (inner iterations) required during the design process or the number of equivalent analyses.

In calculations presented here, Eq. (6) has been used to obtain an approximation to ϕ^{n+1}, instead of obtaining the precise solution through the application of Eq. (5). The use of the approximate relation reduces the total inner iterative sweeps by the factor $(L+\Delta N)/(L+\Delta N+1)$. Since this factor is generally close to unity, the use of Eq. (6) instead of Eq. (5) does reduce the cost of calculations, but not by a substantial amount. In general, the use of Eq. (5) may be preferred since the use of this equation reduces the number of ϕ array storage requirements from four to two. Previous optimization schemes require at least two ϕ array storage spaces.

The scheme presented here is applicable to optimization problems without constraints. It is possible to extend the scheme to optimization problems with constraints. This will require evaluation of the constraint functions in addition to the objective function and updating the design parameters in a way that does not violate the constraints. Some optimization problems with constraints, however, may be solved in an approximate manner using the present scheme by including the quantity to be constrained in the definition of the objective function.

Conclusions

An efficient scheme has been presented for solving optimization problems in which the objective function is dependent on the solution of a partial-differential equation. The scheme eliminates the need for an inner-outer iterative procedure. It solves the partial-differential equation only once, thereby reducing the cost of computation to an extent that would allow its use as a practical tool in optimization problems.

The number of iterative sweeps required to solve the optimization problem is $(L+1)N_s\mu$, where the values of μ for all the examples solved herein were within the range $0.49 \le \mu \le 1.02$. The results indicate that the scheme has good convergence properties for the case in which the initial guess for the vector of design parameters is chosen to closely approximate the optimum solution, as well as for the case in which this initial guess is chosen independent of the optimum solution. For a given set of scheme parameters the optimum ΔN value, which leads to the best scheme efficiency, is that value which leads to the simultaneous convergence of ϕ and $\boldsymbol{P}$.

The optimization scheme requires an iterative procedure for solving the partial-differential equation. The procedure used here is the line relaxation procedure; however, the optimization scheme is not limited to a particular procedure.

Acknowledgments

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